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On the existence of compressive solitary waves in compacting media

M Nakayama and D P Mason

Centre for Nonlinear Studies and Department of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa

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Abstract. Necessary conditions for the existence of compressive solitary-wave solutions of a partial differential equation derived by Scott and Stevenson which describes the two-phase fluid flow in a medium compacting under gravity are derived. It is shown that for compressive solitary-wave solutions to exist which satisfy certain boundary conditions it is necessary that $n = m > 1$ where n and m are the exponents in power laws relating the permeability of the medium and the viscosity of the solid matrix, respectively, to the voidage. The effect of the value of the exponents n and m on the shape of the solitary wave is investigated by using existing analytical solutions and new numerical solutions.

1. Introduction

The third-order nonlinear partial differential equation

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial z}\left[\phi^n\left(1 - \frac{\partial}{\partial z}\left(\frac{1}{\phi^m}\,\frac{\partial \phi}{\partial t}\right)\right)\right] = 0 \tag{1}$$

was derived by Scott and Stevenson [1, 2] and independently for $m = 0$ by Richter and McKenzie [3] and Barcilon and Richter [4], in order to describe the one-dimensional migration of melt through the Earth's mantle. The melt and solid matrix of the mantle are modelled as two immiscible fully connected viscous fluids of constant but different densities. In equation (1), $\phi(z, t)$ is the voidage or volume fraction of melt and n and m are the exponents in power laws relating the permeability of the medium, K, and the bulk and shear viscosities of the solid matrix, ξ and η, to ϕ:

$$K = K_0\phi^n \qquad \xi = \frac{\xi_0}{\phi^m} \qquad \eta = \frac{\eta_0}{\phi^m} \tag{2}$$

where K_0, ξ_0 and η_0 are constants. Scott and Stevenson [1,2] suggest that n lies in the physical range 2–5 and that m lies in the physical range 0–1. We will assume that $n \geqslant 0$ and $m \geqslant 0$. The voidage is normalized by division by the background voidage, ϕ_0, the variable z is the vertical coordinate measured positive upwards and made dimensionless by division by the compaction length, δ_c, defined by

$$\delta_c = \left(\frac{K_0\phi_0^{n-m}\left(\xi_0 + \frac{4}{3}\,\eta_0\right)}{\mu}\right)^{1/2} \tag{3}$$

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where μ is the shear viscosity of the melt, and the variable t is the time made dimensionless by division by the characteristic time, t_0, defined by

$$t_0 = \left(\frac{\mu \left(\xi_0 + \frac{4}{3} \eta_0 \right)}{K_0 \phi_0^{n+m-2}} \right)^{1/2} \frac{1}{g \Delta\rho} \tag{4}$$

where g is the acceleration due to gravity and $\Delta\rho > 0$ is the difference between the density of the solid matrix and the density of the melt. In the derivation of (1) it is assumed that the background voidage $\phi_0 \ll 1$.

Rarefactive solitary-wave solutions of (1) in which $1 \leqslant \phi \leqslant \Psi$, where $\Psi - 1$ is the amplitude of the solitary wave, have been derived by several authors for a variety of values of n and m [1–9]. It can be shown that for a rarefactive solitary-wave solution to exist it is necessary that $n > 1$ [1, 9]. Recently, Nakayama and Mason [10] derived exact compressive solitary-wave solutions of (1), in which $0 \leqslant \phi \leqslant 1$, for the cases $n = m = 2$ and $n = m = \frac{3}{2}$ by imposing the boundary conditions introduced by Zabusky [11] and Jeffrey and Kakutani [12] for the modified Korteweg–de Vries equation. In this communication we will derive necessary conditions on n and m for compressive solitary-wave solutions of (1) to exist subject to the boundary conditions of Zabusky, and Jeffrey and Kakutani. We will also investigate the effect of the value of n and m on the shape of the compressive solitary wave.

An outline of the paper is as follows. The necessary condition, $n > 1$, for the existence of compressive solitary waves is derived in section 2. This condition depends only on the permeability of the medium. In section 3 the necessary condition, $n = m$, for the existence of compressive solitary waves is derived. This condition depends on the viscosity of the solid matrix through the exponent m as well as on the permeability of the medium through the exponent n. The effect of the value of n and m on the shape of the compressive solitary wave is investigated in section 4 by considering analytical and numerical solutions for specific integer and half-integer values of $n = m > 1$. Finally, concluding remarks are made in section 5.

2. Effect of permeability on existence of compressive solitary waves

We will first derive some general results. We look for a travelling solitary-wave solution of (1) of the form

$$\phi(z, t) = \psi(\zeta) \qquad \zeta = z - ct \tag{5}$$

where the constant c is the dimensionless speed of the solitary wave. Equation (1) may be integrated once with respect to ζ and using the identity [1]

$$\frac{\mathrm{d}}{\mathrm{d}\zeta} \left(\frac{1}{\psi^m} \frac{\mathrm{d}\psi}{\mathrm{d}\zeta} \right) = \frac{1}{2} \psi^m \frac{\mathrm{d}}{\mathrm{d}\psi} \left(\frac{1}{\psi^{2m}} \left(\frac{\mathrm{d}\psi}{\mathrm{d}\zeta} \right)^2 \right) \tag{6}$$

it may be integrated once with respect to ψ to give

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}\zeta} \right)^2 = f(\psi) \tag{7}$$

where

$$f(\psi) = \frac{2\psi^{2m}}{c}\left(B - \int^{\psi} \frac{(x^n - cx - A)}{x^{n+m}}\,\mathrm{d}x\right) \tag{8}$$

provided $c \neq 0$, and where A and B are constants. The background state is $\psi = 1$. In order to obtain the three constants, A, B and c, the following three boundary conditions are imposed:

$$f(1) = 0 \tag{9}$$

$$\frac{\mathrm{d}f}{\mathrm{d}\psi}(1) = 0 \tag{10}$$

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\psi^2}(1) = 0. \tag{11}$$

From equations (7) and (6) with $m = 0$, the boundary conditions (9) and (10) are equivalent to the boundary conditions

$$\psi = 1: \qquad \frac{\mathrm{d}\psi}{\mathrm{d}\zeta} = 0 \qquad \frac{\mathrm{d}^2\psi}{\mathrm{d}\zeta^2} = 0 \tag{12}$$

and they are also used to obtain rarefactive solitary-wave solutions [1]. The boundary condition (11) is the condition introduced by Zabuski [11] and Jeffrey and Kakutani [12]. When the boundary conditions (9) to (11) are imposed on (8) the following three equations for A, B and c are obtained:

$$B - \int^{1} \frac{(x^n - cx - A)\mathrm{d}x}{x^{n+m}} = 0 \tag{13}$$

$$c + A - 1 = 0 \tag{14}$$

$$c = n. \tag{15}$$

Thus $c = n$ and therefore for a travelling wave solution to exist it is necessary that $n \neq 0$. In the following we therefore suppose that $n \neq 0$. When $n \neq 0$, (8) becomes

$$f(\psi) = \frac{2\psi^{2m}}{n}\int_{\psi}^{1} \frac{(x^n - nx + n - 1)}{x^{n+m}}\,\mathrm{d}x. \tag{16}$$

We now prove that for a compressive solitary-wave solution of (1) satisfying the boundary conditions (9)–(11) to exist, it is necessary that $n > 1$. From (7) it follows that solitary-wave solutions exist between the non-negative real zeros of $f(\psi)$ provided that $f(\psi) > 0$ between these zeros. Now $f(1) = 0$ and for a compressive solitary wave superimposed on the background state, $\psi \leqslant 1$ (physically, it is also necessary that $\psi \geqslant 0$ because voidage cannot become negative). Thus for a compressive solitary-wave solution to exist it is necessary that $f(\psi) > 0$ for $\psi < 1$ in the neighbourhood of $\psi = 1$. But in the graph of $f(\psi)$ against ψ the point $\psi = 1$ on the ψ-axis is a point of inflexion with a horizontal tangent because $f(1) = 0$, $f'(1) = 0$ and $f''(1) = 0$, where the dash denotes differentiation with respect to the argument. It is therefore necessary that in the neighbourhood of $\psi = 1$ the graph of $f(\psi)$ against ψ be concave up $\big(f''(\psi) > 0\big)$ for $\psi < 1$ and concave down $\big(f''(\psi) < 0\big)$ for $\psi > 1$. Thus $f''(\psi)$ must be a decreasing function of ψ in the neighbourhood of $\psi = 1$ and therefore it is necessary that

$$\frac{d^3 f}{d\psi^3}(1) \leqslant 0. \tag{17}$$

If $f'''(1) = 0$ further investigation is required to determine if a compressive solitary-wave solution actually exists. If $f'''(1) > 0$, a compressive solitary-wave solution superimposed on the background state $\psi = 1$ and satisfying the boundary conditions (9)–(11) does not exist. The condition (17) is illustrated in figure 1. Now, it can be verified that

$$f'''(1) = -2(n-1) \tag{18}$$

and therefore for a compressive solitary-wave solution to exist it is necessary that $n \geqslant 1$. Consider the case $n = 1$ for which $f'''(1) = 0$. When $n = 1$, (7) and (16) yield

$$\frac{d\psi}{d\zeta} = 0 \tag{19}$$

and therefore $\psi =$ constant, which is not a solitary wave. Hence for a compressive solitary-wave solution to exist which satisfies the boundary conditions (9)–(11), it is necessary that $n > 1$. A solution of this kind does not exist if $n \leqslant 1$.

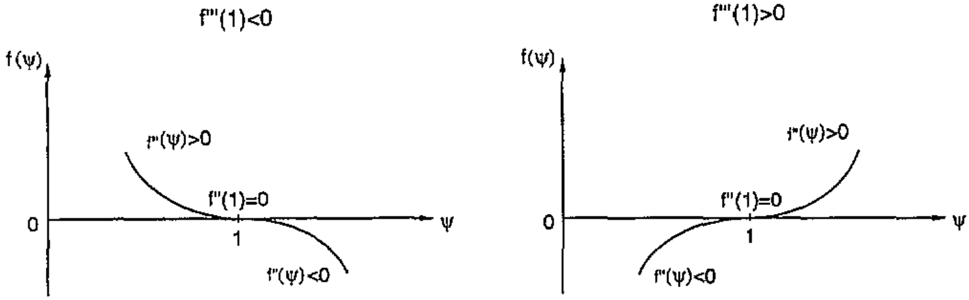

Figure 1. Behaviour of $f(\psi)$ in the neighbourhood of $\psi = 1$ for $f'''(1) < 0$ and $f'''(1) > 0$.

The necessary condition derived in this section depends only on n and is independent of m. It therefore depends only on the permeability of the medium and demonstrates the important part played in the theory by the power law (2) relating the permeability to the voidage. Since a solution does not exist for $n = 0$ the permeability must depend on the voidage for a compressive solitary wave, of the kind considered here, to exist.

3. Effect of matrix viscosity on existence of compressive solitary waves

We first show that when a compressive solitary-wave solution exists the range of ψ extends to $\psi = 0$. We will then obtain the second necessary condition for the existence of a compressive solitary-wave solution by considering the behaviour of $f(\psi)$ as $\psi \to 0$.

We denote the numerator of the integrand of the integral in (16) by $G(x)$,

$$G(x) = x^n - nx + n - 1\,. \tag{20}$$

Then

$$G'(x) = n(x^{n-1} - 1) \tag{21}$$

$$G''(x) = n(n-1)x^{n-2} \tag{22}$$

and

$$G(1) = 0 \qquad G'(1) = 0\,. \tag{23}$$

Now, when $n > 1$, $G''(x) > 0$ for $0 < x \leqslant 1$. Thus $G(x)$ is concave up for $0 < x \leqslant 1$ and since also $G(1) = 0$ and $G'(1) = 0$ it follows that $G(x) > 0$ for $0 \leqslant x < 1$. Thus from (16), $f(\psi) > 0$ for $0 < \psi < 1$ and the range of the solitary wave extends to $\psi = 0$.

A solitary-wave solution may be identified by the behaviour of $f(\psi)$ near its zeros [13]. A simple zero will correspond to a crest or a trough while a double zero, or a triple zero as in the solution considered here, will give an asymptotic tail to ψ near the background state. A compressive solitary-wave solution will therefore correspond to a positive solution $f(\psi)$ between the triple zero of $f(\psi)$ at the background state $\psi = 1$ and a *simple* zero at the trough $\psi = 0$. Thus, for a compressive solitary-wave solution,

$$f(\psi) = \psi F(\psi) \tag{24}$$

where $F(\psi) > 0$ for $0 \leqslant \psi < 1$. The solution exists as $\psi \to 0$ because (7) is a variables separable first-order differential equation and the integral

$$\int^{\psi} \frac{\mathrm{d}\psi}{\psi^{1/2}[F(\psi)]^{1/2}} \tag{25}$$

is convergent as $\psi \to 0$. It also follows from the identity (6) with $m = 0$ that

$$\left.\frac{\mathrm{d}^2\psi}{\mathrm{d}\zeta^2}\right|_{\psi=0} = \tfrac{1}{2}\, f'(0) = \tfrac{1}{2}\, F(0) > 0 \tag{26}$$

so that $\psi(\zeta)$ is indeed a minimum when $\psi = 0$.

We now therefore consider the behaviour of $f(\psi)$ as $\psi \to 0$. When (16) is integrated there are three special cases which introduce logarithms, namely $n + m = 1$, $n + m = 2$ and $m = 1$. But since it is necessary that $n > 1$ and since we assume that $m \geqslant 0$ it follows that $n + m > 1$. The case $n + m = 1$ is therefore not considered. Also, since $n > 1$, the cases $n + m = 2$ and $m = 1$ cannot be satisfied simultaneously. Thus by integrating (16) it can be verified that

$$f(\psi) = \frac{2(n-1)\psi^{m-n+1}}{n(n+m-1)} \begin{cases} \left(1 + \mathrm{O}(\psi) + \mathrm{O}\left(\psi^{n+m-1}\right)\right) & \text{if} \quad n+m \neq 2 \quad \text{and} \quad m \neq 1 & (27) \\ (1 + \mathrm{O}(\psi \ln \psi)) & \text{if} \quad n+m = 2 & (28) \\ (1 + \mathrm{O}(\psi)) & \text{if} \quad m = 1 & (29) \end{cases}$$

as $\psi \to 0$. Thus $f(\psi)$ has a simple zero at $\psi = 0$ provided $n = m$. We see also from (26) that when $n = m > 1$,

$$\left.\frac{\mathrm{d}^2\psi}{\mathrm{d}\zeta^2}\right|_{\psi=0} = \frac{(n-1)}{n(2n-1)} > 0 \tag{30}$$

verifying that ψ attains a minimum value when $\psi = 0$.

The condition $n = m$ depends on the bulk and shear viscosities of the solid matrix through the exponent m as well as on the permeability of the medium through the exponent n. This condition must be combined with the condition $n > 1$ to give the necessary condition $n = m > 1$ for the existence of a compressive solitary-wave solution of the kind considered here. Since it is necessary that $m > 1$, the viscosity of the solid matrix must depend on the voidage for the compressive solitary-wave solution to exist.

4. Solutions for specific values of n and m

Finally, we will compare briefly some solutions for compressive solitary waves with $n = m > 1$. When $n = m$ the characteristic length, δ_c, defined by (3), which is used to make ζ dimensionless, does not depend on n and m. Thus δ_c can still be used as the characteristic length when comparing solutions with different values of $n = m$.

For each value of $n = m$ considered the solution will be reduced to the evaluation of an integral. Except for two cases, the integral will be evaluated numerically. The IMSL subroutine DQDAGS, which was designed to integrate functions which have endpoint singularities, will be used [14]. The performance of the subroutine on functions which are well behaved at the end points is also quite good. The subroutine subdivides the interval of integration and uses a 21-point Gauss–Kronrod rule to estimate the integral over each subinterval.

Consider first the case in which n and m are positive integers such that $n = m \geqslant 2$. Then (16) integrates to give

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}\zeta}\right)^2 = \psi R(\psi) \tag{31}$$

where $R(\psi)$ is a polynomial in ψ of degree $2n - 1$ defined by

$$R(\psi) = \frac{1}{n(n-1)(2n-1)}\left[-n\psi^{2n-1} + 2(2n-1)\psi^n - n(2n-1)\psi + 2(n-1)^2\right]. \tag{32}$$

By Descartes' rule of signs [15], the equation $R(\psi) = 0$ cannot have more positive roots than there are changes of sign from + to − and from − to + in the coefficients of $R(\psi)$. Thus $R(\psi) = 0$ cannot have more than three positive roots and since $R(1) = R'(1) = R''(1) = 0$, it has exactly three positive roots which are at $\psi = 1$, in agreement with the boundary conditions and the general theory. Thus

$$R(\psi) = (1-\psi)^3 P(\psi) \tag{33}$$

where $P(\psi)$ is a polynomial in ψ of degree $2n - 4$ which has no positive zeros and $(\mathrm{d}\psi/\mathrm{d}\zeta)^2 > 0$ for $0 < \psi < 1$. The solitary-wave solution corresponds to the positive solution $(\mathrm{d}\psi/\mathrm{d}\zeta)^2$ between the triple zero at $\psi = 1$ and the simple zero at $\psi = 0$; $(\mathrm{d}\psi/\mathrm{d}\zeta)^2$

is negative and $d\psi/d\zeta$ is therefore imaginary, for $\psi > 1$ and for $\psi < 0$ in the neighbourhood of $\psi = 0$. If we choose $\zeta = 0$ at $\psi = 0$ then the solution can be written implicitly in the form

$$\zeta = \pm \int_0^{\psi} \frac{dx}{x^{1/2}(1-x)^{3/2}\left[P(x)\right]^{1/2}} \tag{34}$$

where for

$$n = m = 2 \qquad P(x) = \tfrac{1}{3} \tag{35}$$

$$n = m = 3 \qquad P(x) = \tfrac{1}{30}\,(3x^2 + 9x + 8) \tag{36}$$

$$n = m = 4 \qquad P(x) = \tfrac{1}{42}\,(2x^4 + 6x^3 + 12x^2 + 13x + 19) \tag{37}$$

$$n = m = 5 \qquad P(x) = \tfrac{1}{180}\,(5x^6 + 15x^5 + 30x^4 + 50x^3 + 57x^2 + 51x + 32)\,. \tag{38}$$

The function ψ is an even function of ζ and therefore symmetric with respect to the trough at $\psi = 0$. When $n = m = 2$, equation (34) can be integrated to give [10]

$$\psi = 1 - \frac{12}{12 + \zeta^2}\,. \tag{39}$$

The compressive solitary-wave solution (39) tends *algebraically* to the background state, $\psi = 1$, as $|\zeta| \to \infty$ which is slower than the limiting behaviour of the rarefactive solitary-wave solutions which tend exponentially to the background state [3–5, 9]. For $n = m = 3$, 4 and 5 the integral in (34) was evaluated numerically using the IMSL subroutine DQDAGS. Graphs of the analytical solution (39) for $n = m = 2$ and of the numerical solutions for $n = m = 3$, 4 and 5 are presented in figure 2. The width of the compressive solitary wave at half its depth, W, is given by

$$W = 2\int_0^{1/2} \frac{dx}{x^{1/2}(1-x)^{3/2}[P(x)]^{1/2}}\,. \tag{40}$$

For $n = m = 2$, 3, 4 and 5, $W = 6.928$, 6.947, 7.402 and 7.935, respectively. The width therefore increases slowly as $n = m$ increases. However, the shape of the compressive solitary wave as $n = m$ ranges from 2 to 5 does not vary greatly which indicates that the analytical solution (39) for $n = m = 2$ is reasonably representative of the other three cases for which analytical solutions are not known.

Consider secondly the case $n = m = \frac{1}{2}(2p + 1)$ where $p \geqslant 1$ is a positive integer. Then integration of (16) gives

$$\left(\frac{d\psi}{d\zeta}\right)^2 = \frac{\psi}{p(2p-1)(2p+1)}\left[-(2p+1)\psi^{2p} + 8p\psi^{(2p+1)/2} - 2p(2p+1)\psi + (2p-1)^2\right]. \tag{41}$$

If we let $y = \psi^{1/2}$ then (41) becomes

$$\left(\frac{dy}{d\zeta}\right)^2 = S(y) \tag{42}$$

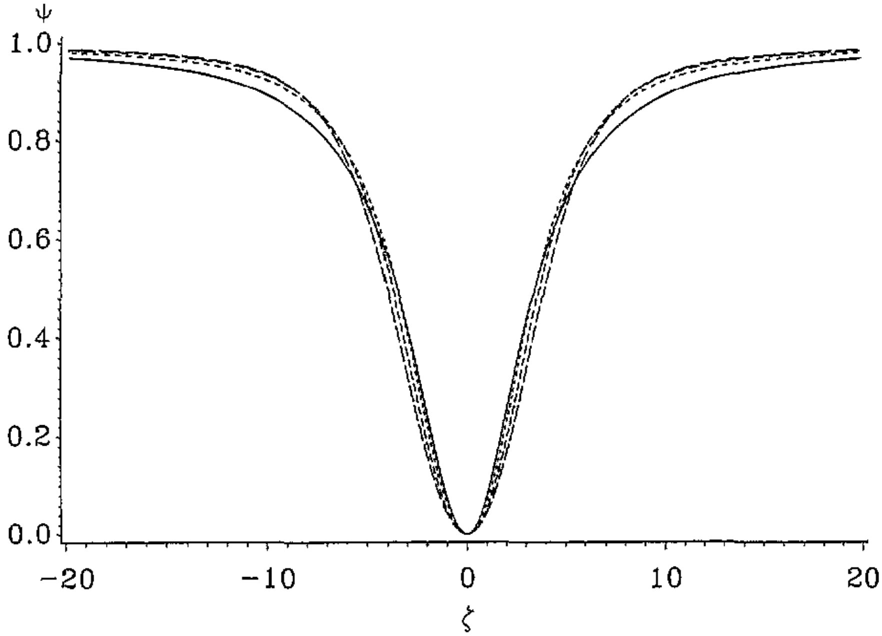

Figure 2. Comparison of compressive solitary-wave solutions: the analytical solution (39) for $n = m = 2$ (————) and the numerical solutions for $n = m = 3$ (- - - - - -), $n = m = 4$ (– – – – –) and $n = m = 5$ (— — —).

where $S(y)$ is a polynomial in y of degree $4p$ defined by

$$S(y) = \frac{1}{4p(2p-1)(2p+1)}\left[-(2p+1)y^{4p} + 8py^{2p+1} - 2p(2p+1)y^2 + (2p-1)^2\right]. \tag{43}$$

By Descartes' rule of signs, $S(y) = 0$ cannot have more than three positive roots and since $S(1) = S'(1) = S''(1) = 0$ it has exactly three positive roots which are at $y = 1$. Thus

$$S(y) = (1-y)^3 Q(y) \tag{44}$$

where $Q(y)$ is a polynomial in y of degree $4p - 3 = 4n - 5$ which has no positive zeros. Also, by Descartes' rule of signs, $S(y) = 0$ cannot have more negative roots than there are changes of sign from + to − and from − to + in the coefficients of $S(-y)$ [15]. Thus $S(y) = 0$ cannot have more than one negative root and since $S(0) = (2p-1)^2 > 0$ and $S(-1) = -16p < 0$, it follows that it has exactly one negative root, which we denote by $y = -\alpha_0$, and that this negative root satisfies $-1 < -\alpha_0 < 0$. The solitary-wave solution corresponds to the positive solution $(\mathrm{d}y/\mathrm{d}\zeta)^2$ between the triple zero at $y = 1$ and the simple zero at $y = -\alpha_0$; $(\mathrm{d}y/\mathrm{d}\zeta)^2$ is negative and $\mathrm{d}y/\mathrm{d}\zeta$ is therefore imaginary, for $y > 1$ and $y < -\alpha_0$. Although $\psi^{1/2}$ takes negative values in part of this range, the voidage, ψ, is always non-negative. If we choose $\zeta = 0$ at $\psi^{1/2} = -\alpha_0$, then the solution may be written in the implicit form

$$\zeta = \pm\int_{-\alpha_0}^{\psi^{1/n}} \frac{\mathrm{d}y}{(1-y)^{3/2}[Q(y)]^{1/2}} \tag{45}$$

Table 1. The root $-\alpha_0$ of the polynomial equation $Q(y) = 0$ where $Q(y)$ is given by (46)–(49) for $n = m = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$ and $\frac{9}{2}$, respectively. The magnitude of the local maximum of the solitary wave is $\psi = \alpha_0^2$.

$n = m$	p	$-\alpha_0$	α_0^2
$\frac{3}{2}$	1	−0.33′	+0.11′
$\frac{5}{2}$	2	−0.613	+0.376
$\frac{7}{2}$	3	−0.728	+0.530
$\frac{9}{2}$	4	−0.790	+0.624

where for

$$n = m = \tfrac{3}{2} \qquad Q(y) = \tfrac{1}{12}\,(3y+1) \tag{46}$$

$$n = m = \tfrac{5}{2} \qquad Q(y) = \tfrac{1}{120}\,(5y^5 + 15y^4 + 30y^3 + 34y^2 + 27y + 9) \tag{47}$$

$n = m = \frac{7}{2}$

$$Q(y) = \tfrac{1}{420}\,(7y^9 + 21y^8 + 42y^7 + 70y^6 + 105y^5 + 123y^4 + 124y^3 + 108y^2 + 75y + 25) \tag{48}$$

$n = m = \frac{9}{2}$

$$Q(y) = \tfrac{1}{1008}\,(9y^{13} + 27y^{12} + 54y^{11} + 90y^{10} + 135y^9 + 189y^8 + 252y^7 + 292y^6 + 309y^5 + 303y^4 + 274y^3 + 222y^2 + 147y + 49) \tag{49}$$

and where, for each value of $n = m$, $-\alpha_0$ is the negative root and only root of the equation

$$Q(y) = 0\,. \tag{50}$$

The values of $-\alpha_0$ are listed in table 1. The function $\psi^{1/2}$ is an even function of ζ and therefore symmetric with respect to the minimum at $\psi^{1/2} = -\alpha_0$. The function ψ is therefore an even function of ζ and symmetric with respect to the local maximum at $\psi = \alpha_0^2$. When $n = m = \frac{3}{2}$, $-\alpha_0 = -\frac{1}{3}$ and (45) can be integrated to give [10]

$$\psi = \left(1 - \frac{12}{9 + \zeta^2}\right)^2 . \tag{51}$$

Equation (51), like (39), describes a solitary wave which tends algebraically to the background state as $|\zeta| \to \infty$. For $n = m = \frac{5}{2}, \frac{7}{2}$ and $\frac{9}{2}$, the integral in (45) was evaluated numerically using IMSL subroutine DQDAGS. Graphs of the analytical solution (51) for $n = m = \frac{3}{2}$ and of the numerical solutions for $n = m = \frac{5}{2}$, $\frac{7}{2}$ and $\frac{9}{2}$ are presented in figure 3. Each solution has two local minima, $\psi = 0$, consistent with (41) and one local maximum, $\psi = \alpha_0^2$. The local maximum increases as p increases. However, since we have proved that $-1 < -\alpha_0 < 0$ for all integers $p \geqslant 1$, the local maximum is always less than unity and the solitary wave is always totally compressive for all integers $p \geqslant 1$. The width of the solitary wave at half its depth, W, is given by

$$W = 2\int_{-\alpha_0}^{1/\sqrt{2}} \frac{\mathrm{d}y}{(1-y)^{3/2}[Q(y)]^{3/2}}\,. \tag{52}$$

For $n = m = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$ and $\frac{9}{2}$, $W = 11.309$, 13.486, 16.046 and 18.254, respectively. The width is greater than when $n = m$ are positive integers and also increases more rapidly as $n = m$ increases.

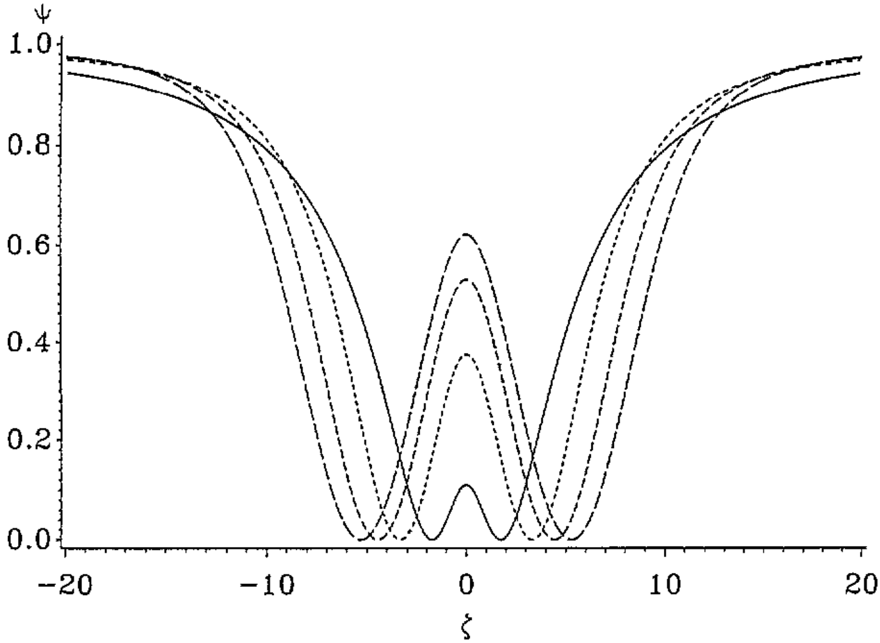

Figure 3. Comparison of compressive solitary-wave solutions: the analytical solution (51) for $n = m = \frac{3}{2}$ (————) and the numerical solutions for $n = m = \frac{5}{2}$ (- - - - - -), $n = m = \frac{7}{2}$ (– – – – –) and $n = m = \frac{9}{2}$ (— — —).

5. Concluding remarks

The values of n and m in the solutions considered here lie in the range $\frac{3}{2}$ to 5. For the Earth's mantle, this is close to the suggested physical range, 2 to 5, for the exponent n but larger than the suggested range, 0 to 1, for the exponent m [1, 2].

When $n = m > 1$ takes on integer values, monotone solitary-wave solutions are obtained. When $n = m > 1$ assumes half-integer values the solitary waves have oscillatory structure although they remain completely compressive. Oscillatory solitary-wave solutions have been derived for other equations. For example, Kawahara [16] found oscillatory solitary-wave solutions, which take both rarefactive and compressive values, for a fifth-order generalized Korteweg–de Vries equation.

We have seen that for compressive solitary-wave solutions of the kind considered here to exist, it is necessary that both the permeability of the medium and the viscosity of the solid matrix depend on the voidage.

One of the assumptions made in the derivation of the partial differential equation (1) is that the background voidage $\phi_0 \ll 1$. It would be of interest to investigate how the results for the existence of compressive solitary-wave solutions depend on ϕ_0 when the approximation that $\phi_0 \ll 1$ is relaxed.

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References

[1] Scott D R and Stevenson D J 1984 *Geophys. Res. Lett.* **11** 1161
[2] Scott D R and Stevenson D J 1986 *J. Geophys. Res.* **91** 9283
[3] Richter F M and McKenzie D 1984 *J. Geology* **92** 729
[4] Barcilon V and Richter F M 1986 *J. Fluid. Mech.* **164** 429
[5] Takahashi D and Satsuma J 1988 *J. Phys. Soc. Japan* **57** 417
[6] Richter F M and Daly S F 1989 *J. Geophys. Res.* **94** 12499
[7] Barcilon V and Lovera O M 1989 *J. Fluid Mech.* **204** 121
[8] Takahashi D, Sachs J R and Satsuma J 1990 *J. Phys. Soc. Japan* **59** 1941
[9] Nakayama M and Mason D P 1992 *Wave Motion* **15** 357
[10] Nakayama M and Mason D P 1991 *Int. J. Non-linear Mech.* **26** 631
[11] Zabusky N J 1967 *Proc. Symp. Nonlinear Partial Differential Equations* ed W Ames (New York: Academic) p 223
[12] Jeffrey A and Kakutani T 1972 *SIAM Rev.* **14** 582
[13] Kichenassamy S and Olver P J 1992 *SIAM J. Math. Anal.* **23** 1141
[14] IMSL Math/Library 1987 *FORTRAN Subroutines for Mathematical Applications* pp 566–8
[15] McArthur N and Keith A 1962 *Intermediate Algebra* (London: Methuen) p 363
[16] Kawahara T 1972 *J. Phys. Soc. Japan* **33** 260